Master thesis

# A Mathematical Model on the Efficiency of Regional Lockdown in Epidemic Dynamics

感染症伝染ダイナミクスにおける
局所的ロックダウンの効果に関する数理モデル

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## Summary

This work considers a mathematical model of epidemic dynamics, focusing on the efficiency of lockdown during the outbreak of a transmissible disease. Lockdown is one of the effective methods to prevent the further spread of an epidemic, though it may bring about economic and social difficulty in the community. Besides the lockdown may lead to some inconvenience in people's life and may cause some psychological disorders. To balance the epidemic control and social activities, the policymaker needs to choose a better policy to take account of a balance of them. In this research project, we consider a simple mathematical model of epidemic dynamics to theoretically discuss the efficiency of lockdown, for which we introduce some different types with respect to the degree of restriction on social activity. The efficiencies are compared to each other according to the endemic size, that is, the number of infective individuals at the endemic equilibrium. In our modeling, we introduce the isolation of infected individuals under the medical treatment in the hospital. By the mathematical analysis on our model, we find that the complete and strong lockdown has the same endemic size, smaller than the weak lockdown. The weak lockdown with minimal restriction on mobility has the lowest efficiency in suppressing the spread of an epidemic.

## Acknowledgements

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# Contents

# 1 Introduction

In addition to the vaccination and medical treatment, the lockdown can be regarded as a basic strategy for the public health to suppress the spread of a transmissible disease in a community. Especially in some regions with poor medical infrastructure and low emergency response capacity, the lockdown could play a role to give the government and decisionmaker a sufficient time to arrange a strategy for controling the epidemic (Lytras and Tsiodras, 2021). The essential role of lockdown is however to reduce the frequency and duration of contacts between individuals in the community. Such a strategy includes closing schools and workplaces, preventing from being outside or gathering, restricting the access to public places (e.g., public transportations), and so on (Oraby et al., 2021). Although such a strict restriction has an important role in suppressing the disease transmission in a community, the economic development must tend to face with great challenges due to the decline in the social activities under it, as seen in the COVID-19 pandemic (Nicola et al., 2020). According to Coccia (2021), it is uncertain whether the long-term lockdown can reduce the number of COVID-19 infected individuals and deaths, and the longer lockdown has a negative impact on the economy. Especially, the economy related to tourism has been severely affected (Wilder-Smith, 2006). Not only the closure of factories and stores has had a great impact on relevant industries (e.g., retail), but also consumers' spending has declined due to restrictive measures and reduced income (Lu et al., 2021). Tonnoir et al. (2021) considered a mathematical model to investigate the optimal investment strategy under the lockdown situation, and derived that it is difficult to ensure both the reduction of regional disparities and economic growth. Furthermore, Ganesan et al. (2021) mentioned that the prolonged lockdown may cause some problems in the physical and mental health. Hence, it is necessary to consider whether a lockdown could allow a balance between the epidemic control and the social activities.

In this work, we consider a simple mathematical model to investigate the efficiency of different types of lockdown according to the endemic size, that is, the number of infective individuals at the endemic equilibrium. In our modeling, we introduce different restrictions on the mobility of individuals and define four types of lockdown: complete lockdown, strong lockdown, weak lockdown type 1, and type 2. The mathematical analysis on our model shows the existence and stability of possible equilibria and compare those four types of lockdown to discuss which type of the lockdown is better according to the endemic size.

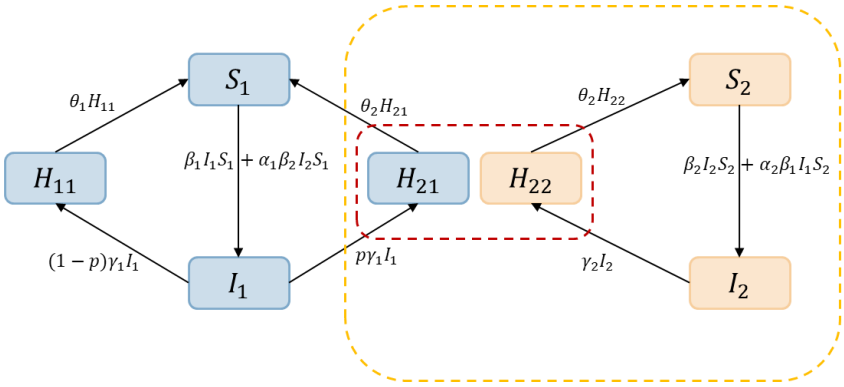


Figure 2.1: Scheme of the epidemic dynamics in our model (2.1).

# 2 Mathematical Modeling

## 2.1 Assumptions

The movement of population must accelerate the spread of an epidemic, which is a fundamental cause of a long-range epidemic transmission. In this work, we consider a simple mathematical model of epidemic dynamics with the following assumptions:

- The disease is not fatal;
- The community is composed of the peripheral area (area 1) and the central area (area 2) with different qualities of the medical treatment for the disease;
- Susceptible individuals of one area can temporarily visit to the other area;
- Some infective individuals of the peripheral area (area 1) can get the medical treatment at the central area (area 2), for example, transported by ambulance;
- Recovered individual becomes susceptible again;
- The population size is constant in each area according to the epidemic dynamics.

## 2.2 Epidemic dynamics

As shown in Figure 2.1, $S_i$ denotes the population density of healthy individuals in area $i$ who can be infected, $I_i$ that of individuals in area $i$ who have been infected and are able to transmit the disease, and $H_{ij}$ that of individuals belonging to area $j$ who are infective

and under the medical treatment in area $i$. We construct the following mathematical model expressed by the system of ordinary differential equations:

$$\begin{aligned}
\frac{dS_1}{dt} &= -\beta_1 I_1 S_1 - \alpha_1\beta_2 I_2 S_1 + \theta_1 H_{11} + \theta_2 H_{21};\\
\frac{dI_1}{dt} &= \beta_1 I_1 S_1 + \alpha_1\beta_2 I_2 S_1 - \gamma_1 I_1;\\
\frac{dH_{11}}{dt} &= (1-p)\gamma_1 I_1 - \theta_1 H_{11};\\
\frac{dH_{21}}{dt} &= p\gamma_1 I_1 - \theta_2 H_{21};\\
\frac{dS_2}{dt} &= -\beta_2 I_2 S_2 - \alpha_2\beta_1 I_1 S_2 + \theta_2 H_{22};\\
\frac{dI_2}{dt} &= \beta_2 I_2 S_2 + \alpha_2\beta_1 I_1 S_2 - \gamma_2 I_2;\\
\frac{dH_{22}}{dt} &= \gamma_2 I_2 - \theta_2 H_{22},
\end{aligned} \tag{2.1}$$

where $\beta_i$ is the infection coefficient in area $i$, which represents the effective infectivity of the transmissible disease. $\alpha_i\beta_j$ is the infection coefficient during the temporary visit to area $j$, which is smaller than $\beta_j$ $(0 < \alpha_i < 1)$. $\gamma_i$ is the treatment rate of the infective in area $i$, and $\theta_i$ is the recovery rate by the medical treatment in area $i$. $p$ is the proportion of infectives belonging to the peripheral area, who get the medical treatment in the central area $(0 \le p \le 1)$. From the assumption, it holds that $S_1 + I_1 + H_{11} + H_{21} = N_1$, $S_2 + I_2 + H_{22} = N_2$ for any time $t$ with positive constants $N_1$ and $N_2$.

With the frequencies $\phi_i = S_i/N_i$, $\psi_i = I_i/N_i$, $\zeta_{ij} = H_{ij}/N_j$, the area-specified basic reproduction numbers $\mathscr{R}_0^r = \beta_1 N_1/\gamma_1$ for the peripheral area and $\mathscr{R}_0^c = \beta_2 N_2/\gamma_2$ for the central area, we can transform the system (2.1) to

$$\begin{aligned}
\frac{d\phi_1}{dt} &= -\mathscr{R}_0^r\gamma_1\psi_1\phi_1 - \mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1 + \theta_1\zeta_{11} + \theta_2\zeta_{21};\\
\frac{d\psi_1}{dt} &= \mathscr{R}_0^r\gamma_1\psi_1\phi_1 + \mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1 - \gamma_1\psi_1;\\
\frac{d\zeta_{11}}{dt} &= (1-p)\gamma_1\psi_1 - \theta_1\zeta_{11};\\
\frac{d\zeta_{21}}{dt} &= p\gamma_1\psi_1 - \theta_2\zeta_{21};\\
\frac{d\phi_2}{dt} &= -\mathscr{R}_0^c\gamma_2\psi_2\phi_2 - \mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2 + \theta_2\zeta_{22};\\
\frac{d\psi_2}{dt} &= \mathscr{R}_0^c\gamma_2\psi_2\phi_2 + \mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2 - \gamma_2\psi_2;\\
\frac{d\zeta_{22}}{dt} &= \gamma_2\psi_2 - \theta_2\zeta_{22},
\end{aligned} \tag{2.2}$$

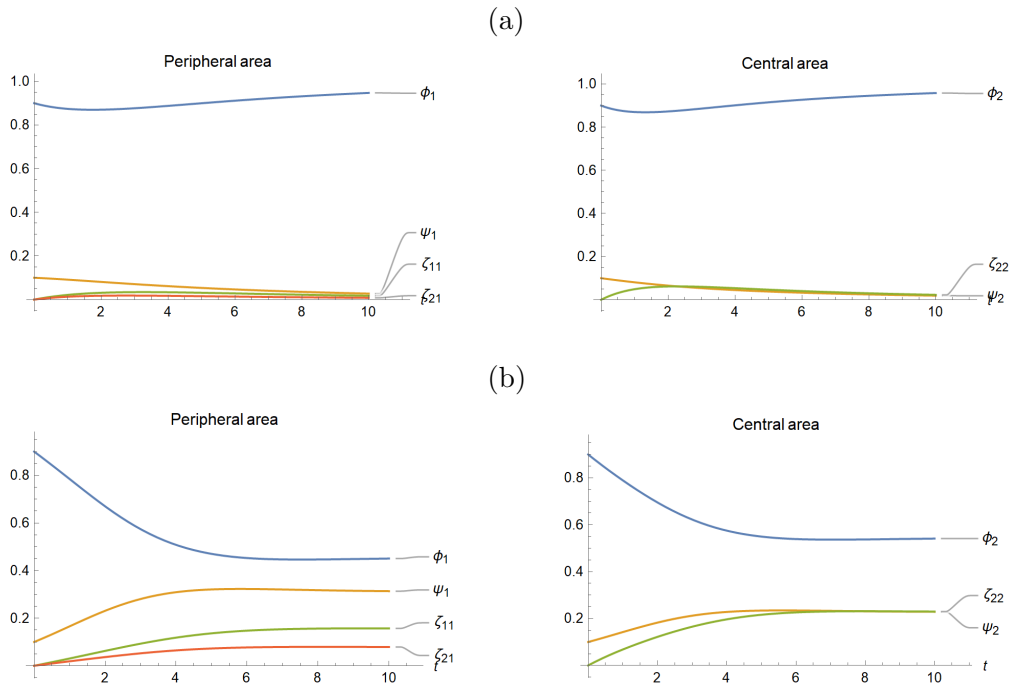


Figure 2.2: Numerical examples of the temporal variation of the frequencies by the system (2.2). $(\mathscr{R}_0^r, \mathscr{R}_0^c)$ = (a) $(0.4, 0.7)$; (b) $(1.5, 1.2)$. Commonly, $\alpha_1 = \alpha_2 = 0.5$; $\theta_1 = 0.6$; $\theta_2 = 0.8$; $\gamma_1 = 0.5$; $\gamma_2 = 0.8$; $p = 0.4$.

where $\phi_1 + \psi_1 + \zeta_{11} + \zeta_{21} = 1$, and $\phi_2 + \psi_2 + \zeta_{22} = 1$. As for the area-specified basic reproduction numbers $\mathscr{R}_0^r$ and $\mathscr{R}_0^c$, we will give the detail later in Section 3. Figure 2.2 shows numerical examples of the temporal variation of the frequencies by the system (2.2) when the disease is extinct and persistent respectively.

## 2.3 Different types of lockdown

Depending on the population size, the severity of pandemic, the economic level, the medical condition, and the living customs in each region, it would be necessary to adopt an appropriate type of lockdown policy. In this work, we consider different levels of restriction on individuals' mobility.

Without the lockdown in our model, the temporary visit of susceptibles is allowed in each area, and infectives of the peripheral area may get the medical treatment in the central area (generally supposing that the central area has a higher quality of the medical treatment than that in the peripheral area). Table 2.1 shows four different types of the lockdown which we introduce in our model. Under the weak lockdown

type 1, only susceptibles of the peripheral area are prohibited to visit the central area. Under the weak lockdown type 2, only susceptibles of the central area are prohibited to visit the peripheral area. Under the strong lockdown, the movement of any susceptibles between two areas is prohibited. For these three types of lockdown, infectives of the peripheral area may get the medical treatment in the central area. In contrast, under the complete lockdown, two areas become fully independent of each other and any movement is prohibited between them, and infectives of the peripheral area cannot get the medical treatment in the central area.

## 3 Basic Reproduction Numbers

The basic reproduction number $\mathscr{R}_0$ is the expected supremum number of secondary cases produced in a totally susceptible population by a single infective individual during the time span of active infectivity (Iannelli and Pugliese, 2015). If $\mathscr{R}_0 < 1$, the number of infectives decreases and the disease will disapper after its invasion in the community. Only if $\mathscr{R}_0 > 1$, the disease could persist after its invasion in the community.

As described in Appendix A for the model (2.1), we can derive the area-specified basic reproduction numbers $\mathscr{R}_0^r = \beta_1 N_1/\gamma_1$ for the peripheral area and $\mathscr{R}_0^c = \beta_2 N_2/\gamma_2$ for the central area respectively. These are the basic reproduction numbers for each area when two areas are fully isolated, that is, the movement of susceptible individuals between them is prohibited. In contrast, the basic reproduction number for the full epidemic dynamics governed by (2.1) can be mathematically defined as

$$\mathscr{R}_0 = \frac{\mathscr{R}_0^r + \mathscr{R}_0^c + \sqrt{(\mathscr{R}_0^r - \mathscr{R}_0^c)^2 + 4\alpha_1\alpha_2\mathscr{R}_0^r\mathscr{R}_0^c}}{2}, \tag{3.1}$$

which is the basic reproduction number for the whole community with the mobility of susceptible individuals. We can easily find that $\mathscr{R}_0 > \mathscr{R}_0^r$ and $\mathscr{R}_0 > \mathscr{R}_0^c$. When

Table 2.1: Different types of lockdown for our model (2.1).

| | $\alpha_1$ | $\alpha_2$ | $p$ |
|---|---|---|---|
| Weak lockdown type 1 | 0 | + | + |
| Weak lockdown type 2 | + | 0 | + |
| Strong lockdown | 0 | 0 | + |
| Complete lockdown | 0 | 0 | 0 |

$\alpha_1\alpha_2 = 0$, that is, under a lockdown introduced in the previous section, we have $\mathscr{R}_0 = \max\{\mathscr{R}_0^r, \mathscr{R}_0^c\}$ from (3.1).

# 4 Existence and Stability of Equilibrium

## 4.1 Disease-free equilibrium

Disease-free equilibrium is defined as an equilibrium state without the disease. For the model (2.2), it becomes $E_0(1,0,0,0,1,0,0)$. By the eigenvalue analysis on the Jacobian matrix for $E_0$, we can obtain the following result on the local stability (Appendix B):

**Theorem 4.1.** *Disease-free equilibrium $E_0(1,0,0,0,1,0,0)$ is unstable if one of the following conditions is satisfied:*

*(i)* $\mathscr{R}_0^r \geq 1$;

*(ii)* $\mathscr{R}_0^c \geq 1$;

*(iii)* $\Big(\frac{1}{\mathscr{R}_0^r} - 1\Big)\Big(\frac{1}{\mathscr{R}_0^c} - 1\Big) < \alpha_1\alpha_2$.

When the mobility of susceptible individuals is sufficiently large, that is, with sufficiently large $\alpha_1\alpha_2$, $E_0$ is unstable with the condition $\mathscr{R}_0^r < 1$ and $\mathscr{R}_0^c < 1$. When $\alpha_1\alpha_2 = 0$, that is, the mobility of susceptible individuals is prohibited for any of two areas, $E_0$ is unstable if and only if the disease persists at least in one of two areas.

## 4.2 Endemic equilibrium

Endemic equilibrium means an equilibrium state at which the number of infectives keeps a positive value for any time $t$. As shown in Appendix C, we can get the following result on the existence of a unique endemic equilibrium $E^*(\phi_1^*, \psi_1^*, \zeta_{11}^*, \zeta_{21}^*, \phi_2^*, \psi_2^*, \zeta_{22}^*)$:

**Lemma 4.1.** *Endemic equilibrium $E^*$ uniquely exists if and only if one of the conditions $(i), (ii)$ and $(iii)$ in Theorem 4.1 is satisfied, independently of which type of lockdown is adopted to the community.*

Especially we can show the global stability of $E^*$ under the complete or strong lockdown, making use of the Lyapunov function (Appendix D):

**Theorem 4.2.** *Under the strong lockdown with $\alpha_1 = \alpha_2 = 0$ or the complete lockdown with $\alpha_1 = \alpha_2 = p = 0$, the endemic equilibrium $E^*$ is globally asymptotically stable when it exists.*

We have not obtained any mathematical result on the global stability of $E^*$ under the weak lockdown or no lockdown. Numerical calculations of the dynamics by (2.2) imply that it would be globally asymptotically stable when it exists.

### 4.3 Endemic size

The proportion of population size in the peripheral area and central area is defined by $\rho := N_1/N_2$. We define here the endemic size as the total number of infective individuals in the community at the endemic equilibrium $E^*$. For our model (2.2), we define it by $\Psi^* := (N_1 + N_2 - S_1^* - S_2^*)/(N_1 + N_2) = 1 - (\rho\phi_1^* + \phi_2^*)/(1+\rho)$. We now designate the endemic sizes under the complete, strong, and weak (type 1 and 2) lockdowns respectively by $\Psi_c^*$, $\Psi_s^*$, $\Psi_{w1}^*$ and $\Psi_{w2}^*$. We can get the following formulas of them from (2.2):

$$\begin{aligned}
\Psi_c^* &= \frac{\rho}{1+\rho}\Big(1 - \frac{1}{\mathscr{R}_0^r}\Big) + \frac{1}{1+\rho}\Big(1 - \frac{1}{\mathscr{R}_0^c}\Big);\\
\Psi_s^* &= \frac{\rho}{1+\rho}\Big(1 - \frac{1}{\mathscr{R}_0^r}\Big) + \frac{1}{1+\rho}\Big(1 - \frac{1}{\mathscr{R}_0^c}\Big);\\
\Psi_{w1}^* &= \frac{\rho}{1+\rho}\Big(1 - \frac{1}{\mathscr{R}_0^r}\Big) + \frac{1}{1+\rho}(1 - \phi_2^*);\\
\Psi_{w2}^* &= \frac{\rho}{1+\rho}(1 - \phi_1^*) + \frac{1}{1+\rho}\Big(1 - \frac{1}{\mathscr{R}_0^c}\Big),
\end{aligned}$$

where $\phi_1^*$ is the smaller root of the following quadratic equation of $x$, which is less than $1/\mathscr{R}_0^r$:

$$\mathscr{R}_0^r\gamma_1\theta_1\theta_2 x^2 - \Big\{(\mathscr{R}_0^r + 1)\gamma_1\theta_1\theta_2 + \mathscr{R}_0^c\alpha_1\psi_2^*\gamma_2\big[\theta_1\theta_2 + (1-p)\gamma_1\theta_2 + p\gamma_1\theta_1\big]\Big\}x + \gamma_1\theta_1\theta_2 = 0$$

with $\psi_2^* = \theta_2[1 - (1/\mathscr{R}_0^c)]/(\theta_2 + \gamma_2)$. $\phi_2^*$ is the smaller root of the following quadratic equation of $x$, which is less than $1/\mathscr{R}_0^c$:

$$\mathscr{R}_0^c\gamma_2\theta_2 x^2 - \big[(\mathscr{R}_0^c + 1)\gamma_2\theta_2 + \mathscr{R}_0^r\alpha_2\psi_1^*\gamma_1(\gamma_2 + \theta_2)\big]x + \gamma_2\theta_2 = 0$$

with $\psi_1^* = \theta_1\theta_2[1 - (1/\mathscr{R}_0^r)]/[\theta_1\theta_2 + (1-p)\gamma_1\theta_2 + p\gamma_1\theta_1]$.

## 5 The Order of Endemic Sizes

Since $\phi_1^* < 1/\mathscr{R}_0^r$ and $\phi_2^* < 1/\mathscr{R}_0^c$, we can easily obtain the order of endemic sizes $\Psi_c^* = \Psi_s^* < \Psi_{w\bullet}^*$. The weak lockdown with minimal restrictions has the least effect on preventing the spread of the epidemic.

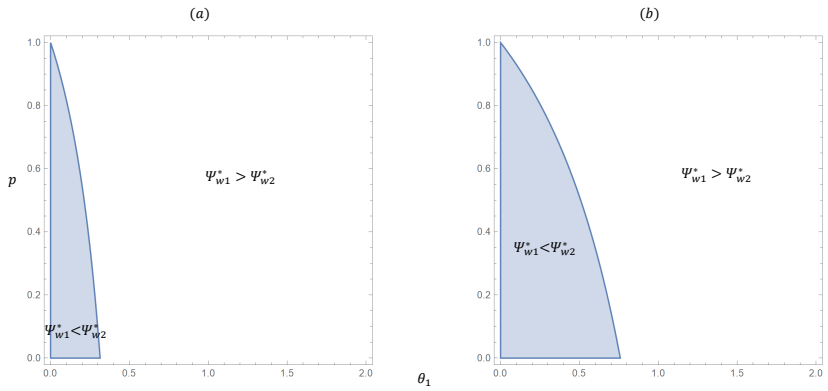


Figure 5.1: Parameter dependence of the order of epidemic sizes for the two types weak lockdown. Numerically drawn with (a) $\theta_2 = 0.6$; (b) $\theta_2 = 1.5$. Commonly, $\rho = 0.5$; $\gamma_1 = 0.5$; $\gamma_2 = 0.9$; $\alpha_1 = \alpha_2 = 0.5$; $\mathscr{R}_0^r = 2$; $\mathscr{R}_0^c = 3$.

### Population distribution

The efficiency of two types of weak lockdown is dependent on the parameter $\rho$, the population distribution of the whole community. $\rho$ is sufficiently small if the population density of the central area is sufficiently larger than that of the peripheral area. In this case, the endemic size is primarily influenced by the endemic size of the central area, we can get that the weak lockdown type 2 is better than type 1. Inversely, if we consider the sufficient large population size of the peripheral area, we can get the opposite result. Therefore, a more efficient weak lockdown is the prohibition of mobility for susceptible individuals from an area of high population density to that of low population density.

### Hospitalization period

In our model, hospitalization period is given by $1/\theta_\bullet$. The extended hospitalization period aims to reduce the pool of susceptible individuals and slow the spread of the epidemic. Referring to Figure 5.1, when hospitalization period of the peripheral area is longer than that of central area, raising the proportion of infected individuals from the peripheral area to get medical treatment in the central area could result in an increase in the endemic size of the peripheral area. Furthermore, for a specific value of $p$, an extended hospitalization period in the central area requires a longer hospitalization period in the peripheral area to ensure that weak lockdown type 1 is better than type 2.

# 6 Discussion

We consider an $SIS+H$ model, where $H$ represents the isolated state in the community. In our model, we assume the community is composed by two areas. The basic reproduction of the whole epidemic system is larger than that of each area, which implies that individuals' mobility could accelerate the spread of transmissible disease. Lockdown is a strategy for preventing mobility which could play a significant role to reduce the spread of the epidemic at the early stage, some countries or regions may implement this policy. In order to achieve a balance between epidemic control and individuals' activity, we need to investigate the efficiency of lockdown policy at the outbreak of epidemic. We consider four types of lockdown by introducing different levels of restriction on individuals' mobility. The mathematical results of the comparison of endemic size for these four types of lockdown indicate that the complete and strong lockdown has the same endemic size, smaller than the weak lockdown. Allowing peripheral infected individuals to get medical treatment in the central area does not have an effect on the endemic size. The weak lockdown with minimal restriction on mobility has the lowest efficiency in suppressing the spread of an epidemic. The population difference between these two areas plays an important role in comparing the endemic size of two types of weak lockdown. For the weak lockdown, more efficient is the prohibition of mobility for susceptible individuals from an area of high population density to that of low population density. When the hospital in the central area has a sufficiently longer isolation period than the peripheral area, free infectives under the strong lockdown are less than those under the complete lockdown.

In this work, we just consider the fixed lockdown policy. For the further research, I am going to study the effect of flexible lockdown, which means the implementation of the mobility restriction changes according to the change of the infection status. This kind of flexible lockdown policy may help to alleviate the economic recession caused by the epidemic to a certain extent.

# Bibliography


Lytras, T. and Tsiodras, S. Lockdowns and the covid-19 pandemic: What is the endgame? *Scandinavian Journal of Public Health*, 49(1): 37–40, 2021. https://doi.org/10.1177/1403494820961293.

Oraby, T., Tyshenko, M.G., Maldonado, J.C., Vatcheva, K., Elsaadany, S., Alali, W.Q., Longenecker, J.C., and Al-Zoughool, M. Modeling the effect of lockdown timing as a covid-19 control measure in countries with differing social contacts. *Scientific reports*, 11(1): 1–13, 2021. https://doi.org/10.1038/s41598-021-82873-2.

Nicola, M., Alsafi, Z., Sohrabi, C., Kerwan, A., Al-Jabir, A., Iosifidis, C., Agha, M., and Agha, R. The socio-economic implications of the coronavirus pandemic (covid-19): A review. *International Journal of Surgery*, 78: 185–193, 2020. https://doi.org/10.1016/j.ijsu.2020.04.018.

Wilder-Smith, A. The severe acute respiratory syndrome: impact on travel and tourism. *Travel Medicine and Infectious Disease*, 4(2): 53–60, 2006. https://doi.org/10.1016/j.tmaid.2005.04.004.

Lu,L., Peng, J., Wu, J., and Lu, Y. Perceived impact of the covid-19 crisis on smes in different industry sectors: Evidence from Sichuan, China. *International Journal of Disaster Risk Reduction*, 55: 102085, 2021. https://doi.org/10.1016/j.ijdrr.2021.102085.

Coccia, M. The relation between length of lockdown, numbers of infected people and deaths of covid-19, and economic growth of countries: Lessons learned to cope with future pandemics similar to covid-19 and to constrain the deterioration of economic system. *Science of The Total Environment*, 775: 145801, 2021. https://doi.org/10.1016/j.scitotenv.2021.145801.

Tonnoir, A., Ciotir, I., Scutariu, A.L., and Dospinescu, O. A model for the optimal investment strategy in the context of pandemic regional lockdown. *Mathematics*, 9(9): 1058, 2021. https://doi.org/10.3390/math9091058.

Ganesan, B., Al-Jumaily, A., Fong, K.N., Prasad, P., Meena, S.K., and Tong, R.K.Y. Impact of coronavirus disease 2019 (covid-19) outbreak quarantine, isolation, and lockdown policies on mental health and suicide. *Frontiers in Psychiatry*, 12(565190): 1–12, 2021. https://doi.org/10.3389/fpsyt.2021.565190.

Iannelli, M. and Pugliese, A. *An Introduction to Mathematical Population Dynamics: Along the Trail of Volterra and Lotka.* Springer, Cham, Switzerland, 2014. https://doi.org/10.1007/978-3-319-03026-5

Brauer, F. and Castillo-Chavez, C. *Mathematical Models in Population Biology and Epidemiology.* Springer, New York, USA, 2012.

## Appendix A. Derivation of the basic reproduction numbers

First, we define the area-specific basic reproduction number $\mathscr{R}_0^r$ for the peripheral area. Let we consider the initial stage of the disease invasion in the peripheral area, at $t = 0$. The number of infective individuals in the peripheral area is sufficiently small. If the disease invasion is successful, the number of infective individuals in the peripheral area increases after it, that is,

$$\left.\frac{dI_1}{dt}\right|_{t=0} = \big(\beta_1 S_1(0) - \gamma_1\big) I_1(0) > 0.$$

This occurs if and only if

$$\frac{\beta_1 S_1(0)}{\gamma_1} > 1. \tag{A.1}$$

If the inequality of (A.1) is inverse, the disease invasion fails, and the number of infective individuals decreases in the peripheral area.

From the biological definition of the basic reproduction number, the disease invasion is successful only if $\mathscr{R}_0^r > 1$, while it fails if $\mathscr{R}_0^r < 1$. Since the basic reproduction number is conceptually defined as the expected number of secondary cases by a single infective individual in a totally susceptible population during the infection, we can define $\mathscr{R}_0^r$ as the supremum of the value $\beta_1 S_1(0)/\gamma_1$ from the condition (A.1) as follows:

$$\mathscr{R}_0^r := \sup_{S_1(0)} \frac{\beta_1 S_1(0)}{\gamma_1} = \frac{\beta_1 N_1}{\gamma_1}.$$

From this definition of $\mathscr{R}_0^r$, we have $dI_1/dt > 0$ only when $\mathscr{R}_0^r > 1$. When $\mathscr{R}_0^r < 1$, we have $dI_1/dt < 0$. The derivation of the area-specified basic reproduction number $\mathscr{R}_0^c$ for the central area is the same as that of $\mathscr{R}_0^r$, and we can define $\mathscr{R}_0^c := \beta_2 N_2/\gamma_2$.

In order to mathematically derive the basic reproduction number $\mathscr{R}_0$ for the community, we use here the method of the next generation matrix (Brauer and Castillo-Chavez,

2012). Firstly, we change the order of equations in (2.2) for a mathematical convenience:

$$\begin{aligned}
\frac{d\psi_1}{dt} &= \mathscr{R}_0^r\gamma_1\psi_1\phi_1 + \mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1 - \gamma_1\psi_1;\\
\frac{d\psi_2}{dt} &= \mathscr{R}_0^c\gamma_2\psi_2\phi_2 + \mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2 - \gamma_2\psi_2;\\
\frac{d\phi_1}{dt} &= -\mathscr{R}_0^r\gamma_1\psi_1\phi_1 - \mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1 + \theta_1\zeta_{11} + \theta_2\zeta_{21};\\
\frac{d\phi_2}{dt} &= -\mathscr{R}_0^c\gamma_2\psi_2\phi_2 - \mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2 + \theta_2\zeta_{22};\\
\frac{d\zeta_{11}}{dt} &= (1-p)\gamma_1\psi_1 - \theta_1\zeta_{11};\\
\frac{d\zeta_{21}}{dt} &= p\gamma_1\psi_1 - \theta_2\zeta_{21};\\
\frac{d\zeta_{22}}{dt} &= \gamma_2\psi_2 - \theta_2\zeta_{22}.
\end{aligned} \tag{A.2}$$

Next we decompose the above system as the form

$$\frac{dX}{dt} = F(X) - V(X),$$

where $X = (\psi_1(t), \psi_2(t), \phi_1(t), \phi_2(t), \zeta_{11}(t), \zeta_{21}(t), \zeta_{22}(t))^T$. $F$ contains only the recruitment terms of the infection, and $V$ does the other factors in (A.2):

$$F := \begin{pmatrix} \mathscr{R}_0^r\gamma_1\psi_1\phi_1+\mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1 \\ \mathscr{R}_0^c\gamma_2\psi_2\phi_2+\mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2 \\ 0\\0\\0\\0\\0 \end{pmatrix}; \quad V := \begin{pmatrix} \gamma_1\psi_1 \\ \gamma_2\psi_2 \\ \mathscr{R}_0^r\gamma_1\psi_1\phi_1+\mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1-\theta_1\zeta_{11}-\theta_2\zeta_{21} \\ \mathscr{R}_0^c\gamma_2\psi_2\phi_2+\mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2-\theta_2\zeta_{22} \\ -(1-p)\gamma_1\psi_1+\theta_1\zeta_{11} \\ -p\gamma_1\psi_1+\theta_2\zeta_{21} \\ -\gamma_2\psi_2+\theta_2\zeta_{22} \end{pmatrix}.$$

The Jacobian matrices of $F$ and $V$ are now obtained as

$$DF(X) := \begin{pmatrix} \mathscr{R}_0^r\gamma_1\phi_1 & \mathscr{R}_0^c\gamma_2\alpha_1\phi_1 & \mathscr{R}_0^r\gamma_1\psi_1+\mathscr{R}_0^c\gamma_2\alpha_1\psi_2 & 0 & 0\ 0\ 0 \\ \mathscr{R}_0^r\gamma_1\alpha_2\phi_2 & \mathscr{R}_0^c\gamma_2\phi_2 & 0 & \mathscr{R}_0^c\gamma_2\psi_2+\mathscr{R}_0^r\gamma_1\alpha_2\psi_1 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \end{pmatrix};$$

$$DV(X) := \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 & 0 \\ \mathscr{R}_0^r\gamma_1\phi_1 & \mathscr{R}_0^c\gamma_2\alpha_1\phi_1 & \mathscr{R}_0^r\gamma_1\psi_1+\mathscr{R}_0^c\gamma_2\alpha_2\psi_2 & 0 & -\theta_1 & -\theta_2 & 0 \\ \mathscr{R}_0^r\gamma_1\alpha_2\phi_2 & \mathscr{R}_0^c\gamma_2\phi_2 & 0 & \mathscr{R}_0^c\gamma_2\psi_2+\mathscr{R}_0^r\gamma_1\alpha_2\psi_1 & 0 & 0 & -\theta_2 \\ -(1-p)\gamma_1 & 0 & 0 & 0 & \theta_1 & 0 & 0 \\ -p\gamma_1 & 0 & 0 & 0 & 0 & \theta_2 & 0 \\ 0 & -\gamma_2 & 0 & 0 & 0 & 0 & \theta_2 \end{pmatrix}.$$

For the disease-free equilibrium $X_0 = (0, 0, 1, 1, 0, 0, 0)^T$, we have

$$DF(X_0) := \begin{pmatrix} \mathscr{R}_0^r\gamma_1 & \mathscr{R}_0^c\gamma_2\alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \mathscr{R}_0^r\gamma_1\alpha_2 & \mathscr{R}_0^c\gamma_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad DV(X_0) := \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 & 0 \\ \mathscr{R}_0^r\gamma_1 & \mathscr{R}_0^c\gamma_2\alpha_1 & 0 & 0 & -\theta_1 & -\theta_2 & 0 \\ \mathscr{R}_0^r\gamma_1\alpha_2 & \mathscr{R}_0^c\gamma_2 & 0 & 0 & 0 & 0 & -\theta_2 \\ -(1-p)\gamma_1 & 0 & 0 & 0 & \theta_1 & 0 & 0 \\ -p\gamma_1 & 0 & 0 & 0 & 0 & \theta_2 & 0 \\ 0 & -\gamma_2 & 0 & 0 & 0 & 0 & \theta_2 \end{pmatrix}.$$

By the upper left $2 \times 2$ block matrix for each of $DF(X_0)$ and $DV(X_0)$, we define

$$\mathcal{F} := \begin{pmatrix} \mathscr{R}_0^r\gamma_1 & \mathscr{R}_0^c\gamma_2\alpha_1 \\ \mathscr{R}_0^r\gamma_1\alpha_2 & \mathscr{R}_0^c\gamma_2 \end{pmatrix}; \quad \mathcal{V} := \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}.$$

Then we can derive the next generation matrix

$$\mathcal{K} = \mathcal{F}\mathcal{V}^{-1} = \begin{pmatrix} \mathscr{R}_0^r & \mathscr{R}_0^c\alpha_1 \\ \mathscr{R}_0^r\alpha_2 & \mathscr{R}_0^c \end{pmatrix}.$$

Since the basic reproduction number $\mathscr{R}_0$ is given by the maximum absolute value of the eigenvalues of $\mathcal{K}$ (Brauer and Castillo-Chavez, 2012), we can get the basic reproduction number $\mathscr{R}_0$ given by (3.1).

## Appendix B. Proof of Theorem 4.1

From (2.2), the Jacobian matrix at the disease-free equilibrium $E_0(1, 0, 0, 0, 1, 0, 0)$ becomes

$$J(1, 0, 0, 0, 1, 0, 0) := \begin{pmatrix} 0 & -\mathscr{R}_0^r\gamma_1 & \theta_1 & \theta_2 & 0 & -\mathscr{R}_0^c\gamma_2\alpha_1 & 0 \\ 0 & (\mathscr{R}_0^r-1)\gamma_1\alpha_2 & 0 & 0 & 0 & \mathscr{R}_0^c\gamma_2 & 0 \\ 0 & (1-p)\gamma_1 & -\theta_1 & 0 & 0 & 0 & 0 \\ 0 & p\gamma_1 & 0 & -\theta_2 & 0 & 0 & 0 \\ 0 & -\mathscr{R}_0^r\gamma_1\alpha_2 & 0 & 0 & 0 & -\mathscr{R}_0^c\gamma_2 & \theta_2 \\ 0 & \mathscr{R}_0^r\gamma_1\alpha_2 & 0 & 0 & 0 & (\mathscr{R}_0^c-1)\gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_2 & -\theta_2 \end{pmatrix}.$$

The characteristic equation of this Jacobian matrix can be obtained as

$$\lambda^2(\lambda+\theta_2)^2(\lambda+\theta_1)\Big\{\big[\lambda-(\mathscr{R}_0^r-1)\gamma_1\big]\big[\lambda-(\mathscr{R}_0^c-1)\gamma_2\big]-\alpha_1\alpha_2\gamma_1\gamma_2\mathscr{R}_0^r\mathscr{R}_0^c\Big\}=0.$$

Then we find $-\theta_1$, and degenerated 0, $-\theta_2$ as the eigenvalues. Besides, we have a quadratic equation given by the last factor in the left side, which determines the other two eigenvalues $\lambda_1$ and $\lambda_2$. Since the discriminant of the quadratic equation is always

positive, $\lambda_1$ and $\lambda_2$ are necessarily real. We can easily find that both $\lambda_1$ and $\lambda_2$ are non-positive if and only if

$$\begin{cases} (\mathscr{R}_0^r - 1)\gamma_1 < 0; \\ (\mathscr{R}_0^c - 1)\gamma_2 < 0; \\ (\mathscr{R}_0^r - 1)(\mathscr{R}_0^c - 1) \geq \alpha_1\alpha_2\mathscr{R}_0^r\mathscr{R}_0^c. \end{cases}$$

If one of the above three conditions is unsatisfied, $\lambda_1$ or $\lambda_2$ is positive. In such a case, the disease-free equilibrium is unstable. The result leads to Theorem 4.1.

## Appendix C. Proof of Lemma 4.1

Consider the existence of an endemic equilibrium $E^*(\phi_1^*, \psi_1^*, \zeta_{11}^*, \zeta_{21}^*, \phi_2^*, \psi_2^*, \zeta_{22}^*)$ with $\psi_1^* > 0$ or $\psi_2^* > 0$. From (2.2), we can derive the following relations of $\psi_1^*$ and $\psi_2^*$:

$$\begin{aligned} \frac{\theta_1\theta_2 + (1-p)\gamma_1\theta_2 + p\gamma_1\theta_1}{\theta_1\theta_2}\psi_1^* =& \frac{\mathscr{R}_0^c\gamma_2\alpha_1\psi_2^* + (\mathscr{R}_0^r - 1)\gamma_1\psi_1^*}{\mathscr{R}_0^r\gamma_1\psi_1^* + \mathscr{R}_0^c\gamma_2\alpha_1\psi_2^*}; \\ \frac{\theta_2 + \gamma_2}{\theta_2}\psi_2^* =& \frac{\mathscr{R}_0^r\gamma_1\alpha_2\psi_1^* + (\mathscr{R}_0^c - 1)\gamma_2\psi_2^*}{\mathscr{R}_0^c\gamma_2\psi_2^* + \mathscr{R}_0^r\gamma_1\alpha_2\psi_1^*}, \end{aligned}$$

that is,

$$\begin{aligned} \psi_2^* &= -\frac{\mathscr{R}_0^r\gamma_1}{\mathscr{R}_0^c\gamma_2\alpha_1}\psi_1^* - \frac{\gamma_1\psi_1^*}{\mathscr{R}_0^c\gamma_2\alpha_1(A\psi_1^* - 1)} = f(\psi_1^*); \\ \psi_1^* &= -\frac{\mathscr{R}_0^c\gamma_2}{\mathscr{R}_0^r\gamma_1\alpha_2}\psi_2^* - \frac{\gamma_2\psi_2^*}{\mathscr{R}_0^r\gamma_1\alpha_2(B\psi_2^* - 1)} = g(\psi_2^*), \end{aligned}$$

where $A := \big[\theta_1\theta_2 + (1-p)\gamma_1\theta_2 + p\gamma_1\theta_1\big]/\big(\theta_1\theta_2\big)$ and $B := \big(\theta_2 + \gamma_2\big)/\theta_2$. The curve of $f(\psi_1)$ has asymptotes $\psi_1 = 1/A$ and

$$\psi_2 = -\frac{\mathscr{R}_0^r\gamma_1}{\mathscr{R}_0^c\gamma_2\alpha_1}\psi_1 - \frac{\gamma_1}{A\mathscr{R}_0^c\gamma_2\alpha_1}.$$

The curve of $g(\psi_2)$ has asymptotes $\psi_2 = 1/B$ and

$$\psi_1 = -\frac{\mathscr{R}_0^c\gamma_2}{\mathscr{R}_0^r\gamma_1\alpha_2}\psi_2 - \frac{\gamma_2}{B\mathscr{R}_0^r\gamma_1\alpha_2}.$$

We have

$$f'(\psi_1) = \frac{\gamma_1}{\mathscr{R}_0^c \gamma_2 \alpha_1}\Big[\frac{1}{(A\psi_1 - 1)^2} - \mathscr{R}_0^r\Big] > 0$$

if and only if

$$\frac{1}{A}\Big(1 - \frac{\sqrt{\mathscr{R}_0^r}}{\mathscr{R}_0^r}\Big) < \psi_1 < \frac{1}{A},\ \frac{1}{A} < \psi_1 < \frac{1}{A}\Big(1 + \frac{\sqrt{\mathscr{R}_0^r}}{\mathscr{R}_0^r}\Big).$$

Further we have

$$f\bigg(\frac{1}{A}\Big(1 - \frac{\sqrt{\mathscr{R}_0^r}}{\mathscr{R}_0^r}\Big)\bigg) = -\frac{\gamma_1\big(1 - \sqrt{\mathscr{R}_0^r}\big)^2}{A\mathscr{R}_0^c \gamma_2 \alpha_1} < 0;\quad f\bigg(\frac{1}{A}\Big(1 + \frac{\sqrt{\mathscr{R}_0^r}}{\mathscr{R}_0^r}\Big)\bigg) = -\frac{\gamma_1\big(1 + \sqrt{\mathscr{R}_0^r}\big)^2}{A\mathscr{R}_0^c \gamma_2 \alpha_1} < 0.$$

When $\psi_1 \to \big(1/A\big)_{-0}$, $f(\psi_1) \to +\infty$, and when $\psi_1 \to \big(1/A\big)_{+0}$, $f(\psi_1) \to -\infty$. Similarly,

$$g'(\psi_2) = \frac{\gamma_2}{\mathscr{R}_0^r \gamma_1 \alpha_2}\Big[\frac{1}{(B\psi_2 - 1)^2} - \mathscr{R}_0^c\Big] > 0$$

if and only if

$$\frac{1}{B}\Big(1 - \frac{\sqrt{\mathscr{R}_0^c}}{\mathscr{R}_0^c}\Big) < \psi_2 < \frac{1}{B},\ \frac{1}{B} < \psi_2 < \frac{1}{B}\Big(1 + \frac{\sqrt{\mathscr{R}_0^c}}{\mathscr{R}_0^c}\Big).$$

Further we have

$$g\bigg(\frac{1}{B}\Big(1 - \frac{\sqrt{\mathscr{R}_0^c}}{\mathscr{R}_0^c}\Big)\bigg) = -\frac{\gamma_2\big(1 - \sqrt{\mathscr{R}_0^c}\big)^2}{B\mathscr{R}_0^r \gamma_1 \alpha_2} < 0;\quad g\bigg(\frac{1}{B}\Big(1 + \frac{\sqrt{\mathscr{R}_0^c}}{\mathscr{R}_0^c}\Big)\bigg) = -\frac{\gamma_2\big(1 + \sqrt{\mathscr{R}_0^c}\big)^2}{B\mathscr{R}_0^r \gamma_1 \alpha_2} < 0.$$

When $\psi_2 \to \big(1/B\big)_{-0}$, $g(\psi_2) \to +\infty$, and when $\psi_2 \to \big(1/B\big)_{+0}$, $g(\psi_2) \to -\infty$. Since $1/A$ and $1/B$ are asymptotes of $f(\psi_1)$ and $g(\psi_2)$ respectively, and both are less than 1, if $\mathscr{R}_0^r > 1$ or $\mathscr{R}_0^c > 1$, two curves $f(\psi_1)$ and $g(\psi_2)$ must have an intersection in the $(\psi_1, \psi_2) = \big((0,1),(0,1)\big)$-plane. We can directly obtain the conclusion from the $(\psi_1, \psi_2)$-plane that the endemic equilibrium $E^*$ exists. If $\mathscr{R}_0^r < 1$ and $\mathscr{R}_0^c < 1$, the endemic equilibrium exists if and only if

$$f'(\psi_1)\big|_{\psi_1=0} < \frac{1}{g'(\psi_2)|_{\psi_2=0}},$$

that is, $\big(1/\mathscr{R}_0^r - 1\big)\big(1/\mathscr{R}_0^c - 1\big) < \alpha_1\alpha_2$. Hence, the conditions for existence of endemic equilibrium $E^*$ are shown in Lemma 4.1.

## Appendix D. Proof of Theorem 4.2

Under the strong lockdown with $\alpha_1 = \alpha_2 = 0$, we can analyze the dynamics for the peripheral area and central area separately. Supposing the endemic equilibrium of peripheral area $E_1^*(\phi_1^*, \psi_1^*, \zeta_{11}^*)$, the system of the epidemic dynamics for the peripheral area can be described from (2.2) as follows:

$$\begin{aligned}
\frac{d\phi_1}{dt} &= -\mathscr{R}_0^r\gamma_1(\psi_1-\psi_1^*)(\phi_1-\phi_1^*) - \mathscr{R}_0^r\gamma_1\psi_1^*(\phi_1-\phi_1^*) - \theta_2(\phi_1-\phi_1^*) \\
&\qquad - (\gamma_1+\theta_2)(\psi_1-\psi_1^*) + (\theta_1-\theta_2)(\zeta_{11}-\zeta_{11}^*); \\
\frac{d\psi_1}{dt} &= \mathscr{R}_0^r\gamma_1(\psi_1-\psi_1^*)(\phi_1-\phi_1^*) + \mathscr{R}_0^r\gamma_1\psi_1^*(\phi_1-\phi_1^*); \\
\frac{d\zeta_{11}}{dt} &= (1-p)\gamma_1(\psi_1-\psi_1^*) - \theta_1(\zeta_{11}-\zeta_{11}^*),
\end{aligned} \tag{D.1}$$

where $\phi_1^* = 1/\mathscr{R}_0^r$;

$$\psi_1^* = \frac{\theta_1\theta_2}{\theta_1\theta_2 + (1-p)\theta_2\gamma_1 + p\theta_1\gamma_1}(1-\phi_1^*);\ \zeta_{11}^* = \frac{(1-p)\theta_2\gamma_1}{\theta_1\theta_2 + (1-p)\theta_2\gamma_1 + p\theta_1\gamma_1}(1-\phi_1^*).$$

Let us define the set $\Omega_1 = \{(\phi_1, \psi_1, \zeta_{11}) \mid \phi_1 \geq 0, \psi_1 \geq 0, \zeta_{11} \geq 0, \phi_1 + \psi_1 + \zeta_{11} \leq 1\}$. For the case of $\theta_1 > \theta_2$, we can find the following Lyapunov equation:

$$\begin{aligned}
V(\phi_1,\psi_1,\zeta_{11}) = &\Big[(\phi_1-\phi_1^*) + (\psi_1-\psi_1^*) + \frac{\theta_1-\theta_2}{\theta_1+\theta_2}(\zeta_{11}-\zeta_{11}^*)\Big]^2 \\
&+ \frac{(\theta_1-\theta_2)\big[\theta_2(2-p)+\theta_1 p\big]}{(\theta_1+\theta_2)^2(1-p)}(\zeta_{11}-\zeta_{11}^*)^2 \\
&+ 2\frac{\big[\theta_2(2-p)+\theta_1 p\big]\gamma_1 + 2\theta_2(\theta_1+\theta_2)}{(\theta_1+\theta_2)\mathscr{R}_0^r\gamma_1}\Big[(\psi_1-\psi_1^*) - \psi_1^*\log\frac{\psi_1}{\psi_1^*}\Big],
\end{aligned} \tag{D.2}$$

which is positive for any $(\phi_1, \psi_1, \zeta_{11}) \neq (\phi_1^*, \psi_1^*, \zeta_{11}^*)$ in $\Omega_1$, and $V(\phi_1^*, \psi_1^*, \zeta_{11}^*) = 0$. Further,

$$\begin{aligned}
\frac{dV(\phi_1,\psi_1,\zeta_{11})}{dt} = &-2\theta_2(\phi_1-\phi_1^*)^2 - 2\Big\{\Big[1-\frac{\theta_1-\theta_2}{\theta_1+\theta_2}(1-p)\Big]\gamma_1+\theta_2\Big\}(\psi_1-\psi_1^*)^2 \\
&- 2\frac{(\theta_1-\theta_2)\big[\theta_1\theta_2+\theta_1^2 p+\theta_2^2(1-p)\big]}{(\theta_1+\theta_2)^2(1-p)}(\zeta_{11}-\zeta_{11}^*)^2
\end{aligned}$$

becomes negative for any $(\phi_1, \psi_1, \zeta_{11}) \neq (\phi_1^*, \psi_1^*, \zeta_{11}^*)$ in $\Omega_1$, and zero for $(\phi_1, \psi_1, \zeta_{11}) = (\phi_1^*, \psi_1^*, \zeta_{11}^*)$.

For the case of $\theta_1 = \theta_2$, we can find the following Lyapunov equation:

$$V(\phi_1, \psi_1, \zeta_{11}) = \big[(\phi_1 - \phi_1^*) + (\psi_1 - \psi_1^*)\big]^2 + \frac{2(2\theta_1 + \gamma_1)}{\mathscr{R}_0^r \gamma_1}\Big[(\psi_1 - \psi_1^*) - \psi_1^* \log\frac{\psi_1}{\psi_1^*}\Big]$$

which is positive for any $(\phi_1, \psi_1, \zeta_{11}) \neq (\phi_1^*, \psi_1^*, \zeta_{11}^*)$ in $\Omega_1$, and $V(\phi_1^*, \psi_1^*, \zeta_{11}^*) = 0$. Further,

$$\frac{dV(\phi_1, \psi_1, \zeta_{11})}{dt} = -2\theta_1(\phi_1 - \phi_1^*)^2 - 2(\gamma_1 + \theta_1)(\psi_1 - \psi_1^*)^2$$

becomes negative for any $(\phi_1, \psi_1, \zeta_{11}) \neq (\phi_1^*, \psi_1^*, \zeta_{11}^*)$ in $\Omega_1$. When $\phi_1 \to \phi_1^*$ and $\psi_1 \to \psi_1^*$, obtain $\zeta_{11}' \to -\theta_1(\zeta_{11} - \zeta_{11}^*)$. Then, denote $\tilde{\zeta_{11}}' = -\theta_1(\tilde{\zeta_{11}} - \zeta_{11}^*)$, since $\zeta_{11}' - \tilde{\zeta_{11}}' \to 0$, obtain $|\zeta_{11} - \tilde{\zeta_{11}}| \to 0$, that is, $\zeta_{11} \to \tilde{\zeta_{11}}$. With $\tilde{\zeta_{11}} \to \zeta_{11}^*$, we get the result that $\zeta_{11} \to \zeta_{11}^*$ when $\phi_1 \to \phi_1^*$ and $\psi_1 \to \psi_1^*$. Hence $dV(\phi_1, \psi_1, \zeta_{11})/dt$ becomes zero for $(\phi_1, \psi_1, \zeta_{11}) = (\phi_1^*, \psi_1^*, \zeta_{11}^*)$.
For the case of $\theta_1 < \theta_2$, considering the reduced system

$$\begin{aligned}
\frac{d\phi_1}{dt} &= -\mathscr{R}_0^r\gamma_1(\psi_1 - \psi_1^*)(\phi_1 - \phi_1^*) - \mathscr{R}_0^r\gamma_1\psi_1^*(\phi_1 - \phi_1^*) - \theta_1(\phi_1 - \phi_1^*) \\
&\qquad - (\gamma_1 + \theta_1)(\psi_1 - \psi_1^*) + (\theta_2 - \theta_1)(\zeta_{21} - \zeta_{21}^*); \\
\frac{d\psi_1}{dt} &= \mathscr{R}_0^r\gamma_1(\psi_1 - \psi_1^*)(\phi_1 - \phi_1^*) + \mathscr{R}_0^r\gamma_1\psi_1^*(\phi_1 - \phi_1^*); \\
\frac{d\zeta_{21}}{dt} &= p\gamma_1(\psi_1 - \psi_1^*) - \theta_2(\zeta_{21} - \zeta_{21}^*),
\end{aligned} \tag{D.3}$$

where $\phi_1^* = 1/\mathscr{R}_0^r$, $\psi_1^* = \theta_2(1 - \phi_1^*)/(\theta_2 + p\gamma_1)$, $\zeta_{21}^* = p\gamma_1(1 - \phi_1^*)/(\theta_2 + p\gamma_1)$.
Let us define the set $\Omega_2 = \{(\phi_1, \psi_1, \zeta_{21}) \mid \phi_1 \geq 0, \psi_1 \geq 0, \zeta_{21} \geq 0, \phi_1 + \psi_1 + \zeta_{21} \leq 1\}$, we can find the following Lyapunov equation:

$$\begin{aligned}
V(\phi_1, \psi_1, \zeta_{21}) = &\Big[(\phi_1 - \phi_1^*) + (\psi_1 - \psi_1^*) + \frac{\theta_2 - \theta_1}{\theta_1 + \theta_2}(\zeta_{21} - \zeta_{21}^*)\Big]^2 \\
&+ \frac{(\theta_2 - \theta_1)\big[\theta_2(1-p) + \theta_1(1+p)\big]}{(\theta_1 + \theta_2)^2 p}(\zeta_{21} - \zeta_{21}^*)^2 \\
&+ 2\frac{\big[\theta_2(1-p) + \theta_1(1+p)\big]\gamma_1 + 2\theta_1(\theta_1 + \theta_2)}{(\theta_1 + \theta_2)\mathscr{R}_0^r\gamma_1}\Big[(\psi_1 - \psi_1^*) - \psi_1^* \log\frac{\psi_1}{\psi_1^*}\Big]
\end{aligned}$$

which is positive for any $(\phi_1, \psi_1, \zeta_{21}) \neq (\phi_1^*, \psi_1^*, \zeta_{21}^*)$ in $\Omega_2$, and $V(\phi_1^*, \psi_1^*, \zeta_{21}^*) = 0$. Further,

$$\frac{dV(\phi_1, \psi_1, \zeta_{21})}{dt} = -2\theta_1(\phi_1 - \phi_1^*)^2 - 2\Big\{\Big[1 - \frac{(\theta_2 - \theta_1)p}{\theta_1 + \theta_2}\Big]\gamma_1 + \theta_1\Big\}(\psi_1 - \psi_1^*)^2 - 2\frac{(\theta_2 - \theta_1)\big[\theta_1\theta_2 + \theta_1^2 p + \theta_2^2(1-p)\big]}{(\theta_1 + \theta_2)^2 p}(\zeta_{21} - \zeta_{21}^*)^2$$

becomes negative for any $(\phi_1, \psi_1, \zeta_{21}) \neq (\phi_1^*, \psi_1^*, \zeta_{21}^*)$ in $\Omega_2$, and zero for $(\phi_1, \psi_1, \zeta_{21}) = (\phi_1^*, \psi_1^*, \zeta_{21}^*)$.
Then, supposing the endemic equilibrium of central area $E_2^*(\phi_2^*, \psi_2^*)$, the system of the epidemic dynamics for the central area can be described from (2.2) as follows:

$$\begin{aligned}
\frac{d\phi_2}{dt} &= -\mathscr{R}_0^c\gamma_2(\psi_2 - \psi_2^*)(\phi_2 - \phi_2^*) - \mathscr{R}_0^c\gamma_2\psi_2^*(\phi_2 - \phi_2^*) - \theta_2(\phi_2 - \phi_2^*) \\
&\quad - (\gamma_2 + \theta_2)(\psi_2 - \psi_2^*); \qquad \text{(D.4)} \\
\frac{d\psi_2}{dt} &= \mathscr{R}_0^c\gamma_2(\psi_2 - \psi_2^*)(\phi_2 - \phi_2^*) + \mathscr{R}_0^c\gamma_2\psi_2^*(\phi_2 - \phi_2^*),
\end{aligned}$$

where $\phi_2^* = 1/\mathscr{R}_0^c$, $\psi_2^* = \theta_2(1 - \phi_2^*)/(\gamma_2 + \theta_2)$. Let us define the set $\Omega_2' = \{(\phi_2, \psi_2) | \phi_2 \geq 0, \psi_2 \geq 0, \phi_2 + \psi_2 \leq 1\}$, we can find the following Lyapunov equation:

$$V(\phi_2, \psi_2) = \big[(\phi_2 - \phi_2^*) + (\psi_2 - \psi_2^*)\big]^2 + \frac{2(2\theta_2 + \gamma_2)}{\mathscr{R}_0^c\gamma_2}\Big[(\psi_2 - \psi_2^*) - \psi_2^* \log\frac{\psi_2}{\psi_2^*}\Big]$$

which is positive for any $(\phi_2, \psi_2) \neq (\phi_2^*, \psi_2^*)$ in $\Omega_2'$, and $V(\phi_2^*, \psi_2^*) = 0$. Further,

$$\frac{dV(\phi_2, \psi_2)}{dt} = -2\theta_2(\phi_2 - \phi_2^*)^2 - 2(\gamma_2 + \theta_2)(\psi_2 - \psi_2^*)^2$$

becomes negative for any $(\phi_2, \psi_2) \neq (\phi_2^*, \psi_2^*)$ in $\Omega_2'$, and zero for $(\phi_2, \psi_2) = (\phi_2^*, \psi_2^*)$. Thus, under the strong lockdown, the endemic equilibrium $E_s^*(\phi_1^*, \psi_1^*, \zeta_{11}^*, \zeta_{21}^*, \phi_2^*, \psi_2^*, \zeta_{22}^*)$ is globally asymptotically stable.

Then, consider the stability of endemic equilibrium under the complete lockdown, the case of $\alpha_1 = \alpha_2 = p = 0$. Supposing the endemic equilibrium of peripheral area $E_{11}^*(\phi_1^*, \psi_1^*)$, the system of the epidemic dynamics for the peripheral area can be described

from (2.2) as follows:

$$\begin{aligned}\frac{d\phi_1}{dt} &= -\mathscr{R}_0^r\gamma_1(\psi_1-\psi_1^*)(\phi_1-\phi_1^*) - \mathscr{R}_0^r\gamma_1\psi_1^*(\phi_1-\phi_1^*) - \theta_1(\phi_1-\phi_1^*)\\ &\quad - (\gamma_1+\theta_1)(\psi_1-\psi_1^*);\\ \frac{d\psi_1}{dt} &= \mathscr{R}_0^r\gamma_1(\psi_1-\psi_1^*)(\phi_1-\phi_1^*) + \mathscr{R}_0^r\gamma_1\psi_1^*(\phi_1-\phi_1^*),\end{aligned}$$

where $\phi_1^* = 1/\mathscr{R}_0^r$, $\psi_1^* = \theta_1(1-\phi_1^*)/(\theta_1+\gamma_1)$. Define the set $\Omega_1' = \{(\phi_1,\psi_1)|\phi_1 \geq 0, \psi_1 \geq 0, \phi_1+\psi_1 \leq 1\}$, we can find the following Lyapunov equation:

$$V(\phi_1,\psi_1) = \left[(\phi_1-\phi_1^*)+(\psi_1-\psi_1^*)\right]^2 + \frac{2(2\theta_1+\gamma_1)}{\mathscr{R}_0^r\gamma_1}\left[(\psi_1-\psi_1^*) - \psi_1^*\log\frac{\psi_1}{\psi_1^*}\right]$$

which is positive for any $(\phi_1,\psi_1) \neq (\phi_1^*,\psi_1^*)$ in $\Omega_1'$, and $V(\phi_1^*,\psi_1^*) = 0$. Further,

$$\frac{dV(\phi_1,\psi_1)}{dt} = -2\theta_1(\phi_1-\phi_1^*)^2 - 2(\gamma_1+\theta_1)(\psi_1-\psi_1^*)^2$$

becomes negative for any $(\phi_1,\psi_1) \neq (\phi_1^*,\psi_1^*)$ in $\Omega_1'$, and zero for $(\phi_1,\psi_1) = (\phi_1^*,\psi_1^*)$. Then, supposing the endemic equilibrium of central area $E_{22}^*(\phi_2^*,\psi_2^*)$, the system of the epidemic dynamics for the central area can be described from (2.2) as follows:

$$\begin{aligned}\frac{d\phi_2}{dt} &= -\mathscr{R}_0^c\gamma_2(\psi_2-\psi_2^*)(\phi_2-\phi_2^*) - \mathscr{R}_0^c\gamma_2\psi_2^*(\phi_2-\phi_2^*) - \theta_2(\phi_2-\phi_2^*)\\ &\quad - (\gamma_2+\theta_2)(\psi_2-\psi_2^*);\\ \frac{d\psi_2}{dt} &= \mathscr{R}_0^c\gamma_2(\psi_2-\psi_2^*)(\phi_2-\phi_2^*) + \mathscr{R}_0^c\gamma_2\psi_2^*(\phi_2-\phi_2^*),\end{aligned}$$

where $\phi_2^* = 1/\mathscr{R}_0^c$, $\psi_2^* = \theta_2(1-\phi_2^*)/(\gamma_2+\theta_2)$. Let us define the set $\Omega_{22}' = \{(\phi_2,\psi_2)|\phi_2 \geq 0, \psi_2 \geq 0, \phi_2+\psi_2 \leq 1\}$, we can find the following Lyapunov equation:

$$V(\phi_2,\psi_2) = \left[(\phi_2-\phi_2^*)+(\psi_2-\psi_2^*)\right]^2 + \frac{2(2\theta_2+\gamma_2)}{\mathscr{R}_0^c\gamma_2}\left[(\psi_2-\psi_2^*) - \psi_2^*\log\frac{\psi_2}{\psi_2^*}\right]$$

which is positive for any $(\phi_2,\psi_2) \neq (\phi_2^*,\psi_2^*)$ in $\Omega_{22}'$, and $V(\phi_2^*,\psi_2^*) = 0$. Further,

$$\frac{dV(\phi_2,\psi_2)}{dt} = -2\theta_2(\phi_2-\phi_2^*)^2 - 2(\gamma_2+\theta_2)(\psi_2-\psi_2^*)^2$$

becomes negative for any $(\phi_2,\psi_2) \neq (\phi_2^*,\psi_2^*)$ in $\Omega_{22}'$, and zero for $(\phi_2,\psi_2) = (\phi_2^*,\psi_2^*)$. Thus, under the complete lockdown, endemic equilibrium $E_c^*$ is globally asymptotically stable.